Algorithm for Sampling Maximal Couplings of Markov Chains

Alan Boles[1], Leila Dahlia[2], Scott McIntyre[3], Bronson Zhou[4]
Project Advisor: Iddo Ben-Ari
[1]University of Tennessee, Knoxville; [2]University of Illinois, Chicago; [3]University of California, Berkeley; [4]University of Texas, Austin

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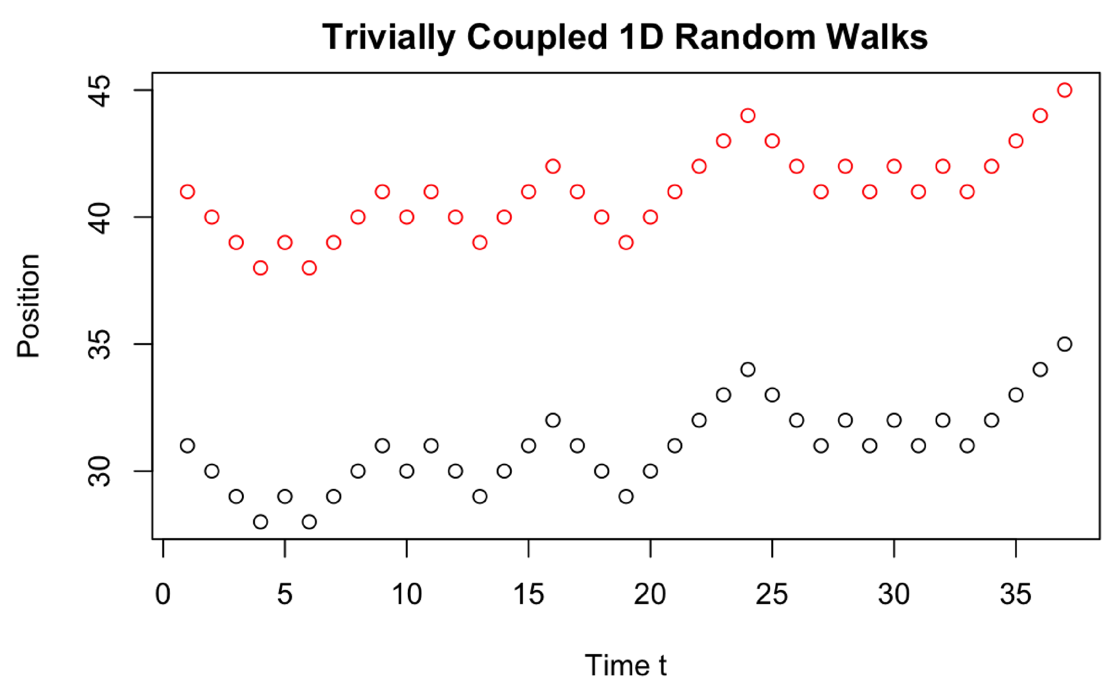
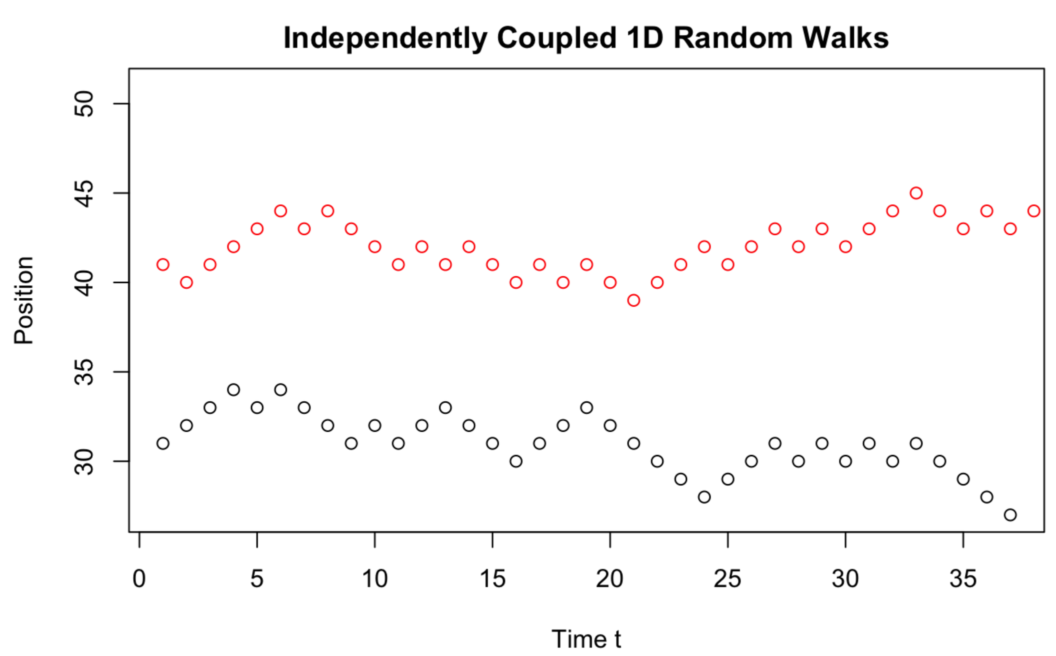
Defining a Maximal Coupling

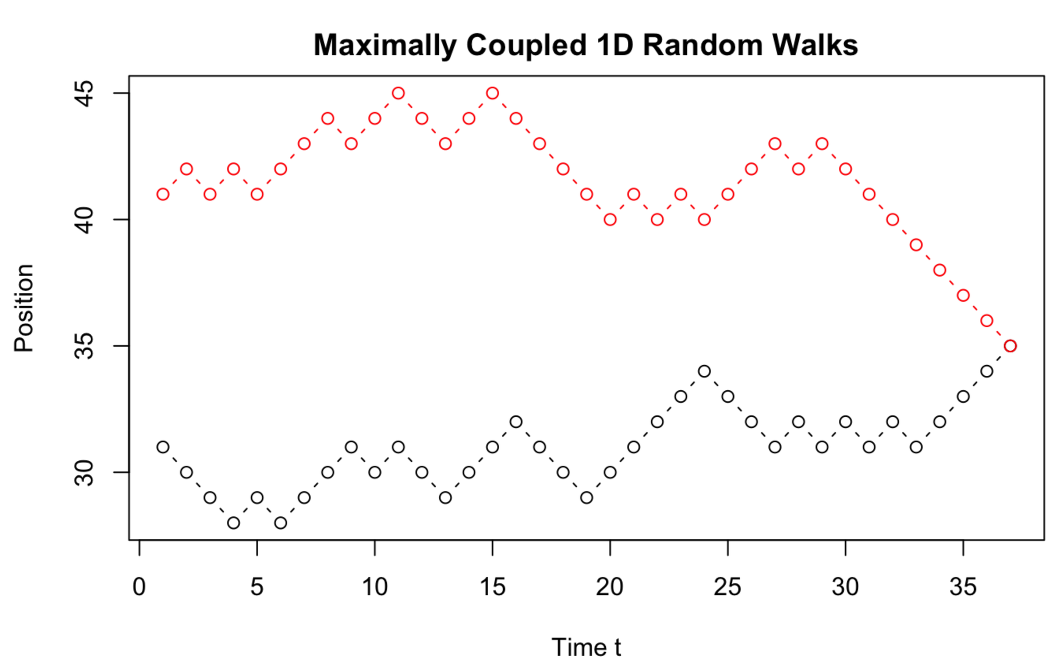
Given two Markov Chains X_t, Y_t with initial distributions μ, ν and shared transition function P, a *coupling* introduces some kind of dependence between X_t, Y_t



Here X_t moves randomly, but Y_t depends on X_t and stays parallel



Here X_t and Y_t are independent; the path of X_t does not affect the path of Y_t.



X_t and Y_t are **maximally coupled**; the dependence maximizes the rate at which they reach the same state.

Constructing a Maximal Coupling

Let T be the first time the paths of X_t, Y_t meet (the coupling time)

Theorem 1. *If X_t, Y_t are maximally coupled, we can recursively compute:*

$$m_t(x) = \mathbb{P}[X_T = Y_T = x, T = t]$$

$$\alpha_{\mu,t}(x) = \mathbb{P}[X_t = x, T > t] \qquad \alpha_{\nu,t}(y) = \mathbb{P}[Y_t = y, T > t]$$

$$C_{\mu,t}(x) = \mathbb{P}[T = t | X_t = x, T \geq t] \qquad C_{\nu,t}(y) = \mathbb{P}[T = t | Y_t = y, T \geq t]$$

Probability that maximally coupled X takes on path $\bar{x} = (x_0, \dots, x_t)$ and $T = t$:

$$\mathbb{P}(X_{[0,t]} = \bar{x}, T = t) = \mu(x_0) C_{\mu,t}(x_t) \prod_{s=0}^{t-1} (1 - C_{\mu,s}(x_s)) P(x_s, x_{s+1}) \tag{1}$$

Time Reversal

Note the substitutions:

$$q_t(y, x) := \frac{\alpha_{\nu,t-1}(x) P(x,y)(1 - C_{\nu,t}(y))}{\alpha_t(y)} \tag{2}$$

$$Q_t(y, x) := \frac{\alpha_{\nu,t-1}(x) P(x,y) C_{\nu,t}(y)}{m_t(y)} \tag{3}$$

This reverses the time in our original path decomposition formula:

$$\mathbb{P}(Y_{[0,t]} = \bar{y} \,|\, T = t, X_t = Y_t = y_t) = Q_t(y_t, y_{t-1}) \prod_{n=1}^{t-1} q_n(y_n, y_{n-1}) \tag{4}$$

Algorithm

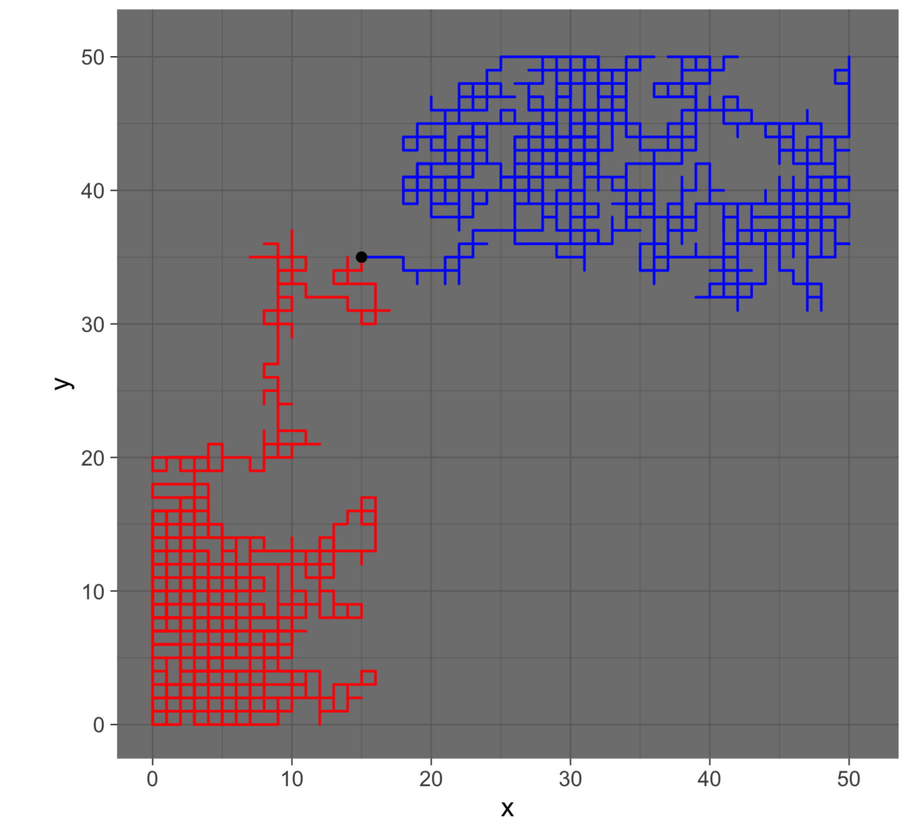
We first sample the X path, and then iterate backward to sample the Y path:

Sample a Maximal Coupling

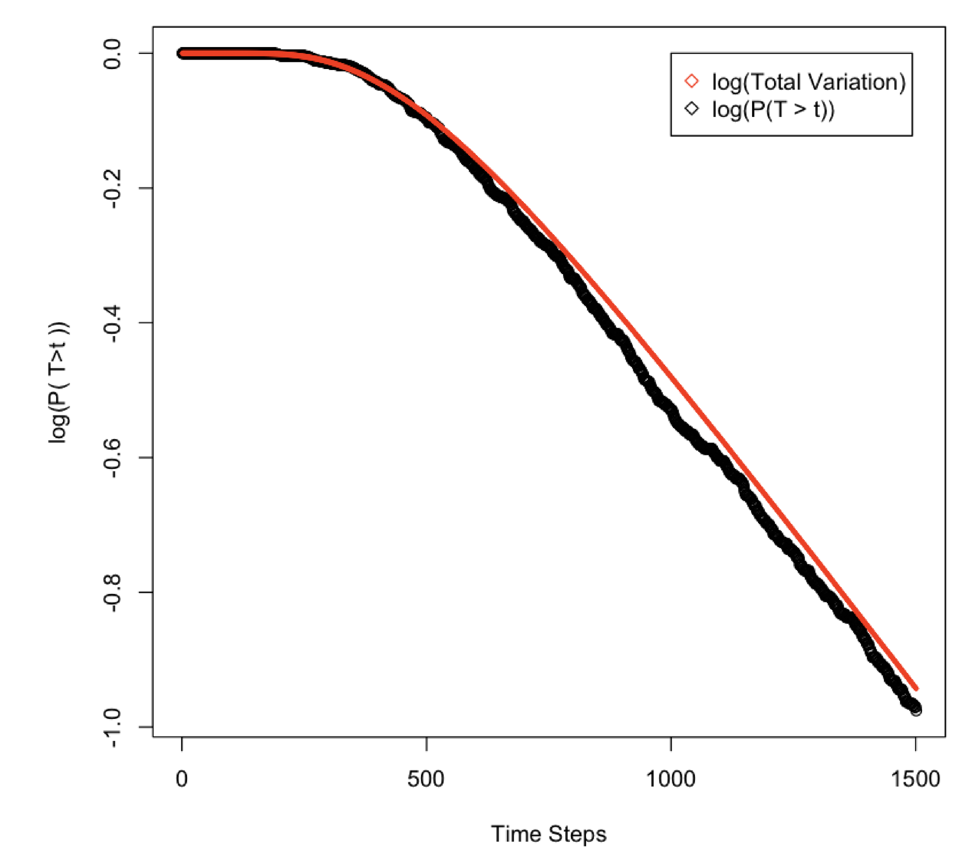
0: **procedure** SAMPLE(P, μ, ν)
0: $t = 0$
0: $X_0 \sim \mu$
0: **while** Uniform$[0,1] < C_{\mu,t}(X_t)$ **do**
0: $X_{t+1} \sim P(X_t, \cdot)$
0: $t = t + 1$
0: **end while**
0: $Y_t = X_t$
0: $Y_{t-1} \sim Q_t(Y_t, \cdot)$
0: $t = t - 1$
0: **while** $t > 0$ **do**
0: $Y_{t-1} \sim q_t(Y_t, \cdot)$
0: $t = t - 1$
0: **end while**

Simulating a Maximal Coupling

Maximally Coupled Walks on a 50×50 Grid



Theoretical TV vs. Empirical TV of Maximally Coupled Walks on a 50×50 Grid



References

References

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